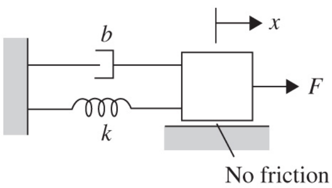
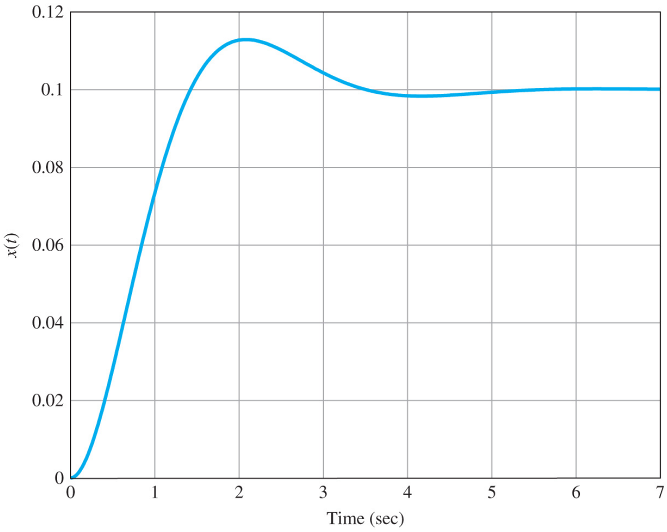
Control Systems : Set 4 : PID (3) - Solutions

Prob 1 | A simple mechanical system is shown in the figure below. The parameters are k = spring constant, b = viscous friction constant and m = mass. A step of 2 Newtons force is applied and the resulting step response is shown below. What are the values of the system parameters k, b, and m?

Hi





The equation of motion for this system is

$$m\ddot{x} + b\dot{x} + kx = F$$

which gives the transfer function

$$\frac{X}{F} = G(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

From the plot, we can estimate a number of properties:

DC gain	$2G(0) = 0.1$	$\rightarrow$	$G(0) = 0.05$
Overshoot	$\frac{x(T_p) - x(\infty)}{x(\infty)} = \frac{0.113 - 0.1}{0.1} = 13\%$	$\rightarrow$	$\zeta = 0.54$
Peak time	$T_p = 2$	$\rightarrow$	$\omega_n \qquad = 1.87$

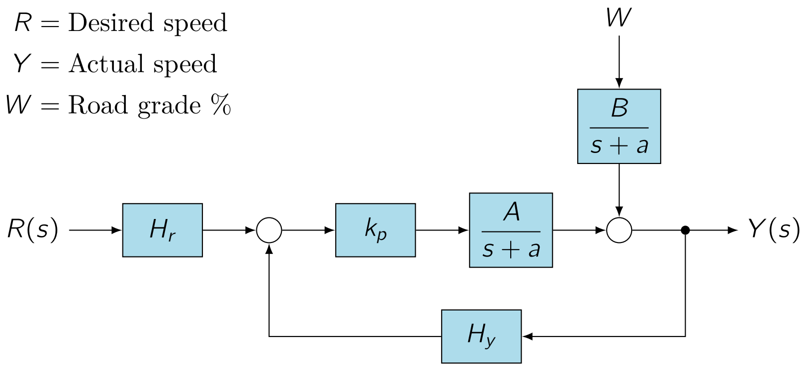
We can now solve for the system parameters

$$G(0) = 0.05 = \frac{1}{k} \quad \rightarrow \quad k = 20$$

$$\omega_n^2 = 3.4969 = \frac{k}{m} = \quad \rightarrow \quad m = 5.72$$

$$2\zeta\omega_n = 2.0196 = \frac{b}{m} \quad \rightarrow \quad b = 11.55$$

Prob 2 | Consider the automobile speed control system depicted in the figure below.



a) Find the transfer functions from $W(s)$ and from $R(s)$ to $Y(s)$.

> Start at Y and work backwards
>
> $$\begin{aligned} Y &= \frac{B}{s+a}W + \frac{k_pA}{s+a}(H_rR - H_yY) \\ (s+a)Y &= BW + k_pAH_rR - k_pAH_yY \\ (s+a+k_pAH_y)Y &= BW + k_pAH_rR \\ Y &= \frac{B}{s+(a+k_pAH_y)}W + \frac{k_pAH_r}{s+(a+k_pAH_y)}R \end{aligned}$$
>
> So we see that the transfer functions are
>
> $$\frac{Y}{W} = \frac{B}{s+(a+k_pAH_y)} \qquad \frac{Y}{R} = \frac{k_pAH_r}{s+(a+k_pAH_y)}$$

b) Assume that the desired speed is a constant reference r_o, so that $R(s) = r_o/s$. Assume that the road is level, so $w(t) = 0$. Compute values of the feedforward gain H_r to guarantee that

$$\lim_{t\to\infty} y(t) = r_o$$

Consider two cases

(i) $H_y = 0$. This is an open-loop feed-forward controller, as there is no feedback.

(ii) $H_y \neq 0$. This is now a closed-loop feedback controller.

> The speed is
>
> $$Y = \frac{k_pAH_r}{s+(a+k_pAH_y)}R = \frac{k_pAH_r}{s+(a+k_pAH_y)}\frac{r_o}{s}$$

We use the Final Value Theorem to compute the steady-state output

$$\begin{aligned} y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) &= \lim_{s\to 0} s\frac{k_pAH_r}{s+(a+k_pAH_y)}\frac{r_o}{s} \\ &= \lim_{s\to 0} \frac{k_pAH_r}{s+(a+k_pAH_y)}r_o \\ &= \frac{k_pAH_r}{a+k_pAH_y}r_o \end{aligned}$$

(i) Case $H_y = 0$

$$y_{ss} = \frac{k_pAH_r}{a}r_o$$

Therefore in order for y_{ss} to equal r_o, we must choose $H_r = \frac{a}{Ak_p}$.

(ii) Case $H_y \neq 0$

$$y_{ss} = \frac{k_pAH_r}{a+k_pAH_y}r_o$$

Therefore in order for y_{ss} to equal r_o, we must choose $H_r = \frac{a+k_pAH_y}{k_pA}$.

c) Now assume that a constant grade disturbance $W(s) = w_o/s$ is present in addition to the reference input. Find the variation in speed Y due to the grade change for both the feed-forward and feedback cases, using the values for H_r computed in part (b). Use your results to explain (i) why feedback control is necessary and (ii) how the gain k_p should be chosen to reduce steady-state error.

From (a) we see that the output will be

$$Y = \frac{B}{s+(a+k_pAH_y)}\frac{w_o}{s} + \frac{k_pAH_r}{s+(a+k_pAH_y)}\frac{r_o}{s}$$

Applying the Final Value Theorem, we get

$$\begin{aligned} y_{ss} &= \lim_{s\to 0} \frac{B}{s+(a+k_pAH_y)}w_o + \frac{k_pAH_r}{s+(a+k_pAH_y)}r_o \\ &= \frac{B}{a+k_pAH_y}w_o + \frac{k_pAH_r}{a+k_pAH_y}r_o \end{aligned}$$

(i) Case $H_y = 0$ and $H_r = \frac{a}{Ak_p}$.

$$\begin{aligned} y_{ss} &= \frac{B}{a+k_pAH_y}w_o + \frac{k_pAH_r}{a+k_pAH_y}r_o \\ &= \frac{B}{a}w_o + r_o \end{aligned}$$

We see that the variation in y_{ss} is $\dfrac{B}{a}$. This value is completely unaffected by the feedforward term, and therefore we cannot change the impact of the disturbance on the output.

(ii) Case $H_y \neq 0$ and $H_r = \frac{a+k_pAH_y}{k_pA}$.

$$
\begin{aligned}
y_{ss} &= \frac{B}{a+k_pAH_y} w_o + \frac{k_pAH_r}{a+k_pAH_y} r_o \\
&= \frac{B}{a+k_pAH_y} w_o + r_o
\end{aligned}
$$

We can now reduce the impact of the disturbance by choosing a larger value for the proportional control term k_p.

d) Assume that $w(t) = 0$ and that the gain A undergoes the perturbation $A + \delta A$. Determine the error in speed due to the gain change for both the feed forward and feedback cases (use the values for H_r derived in (b)). How should the gains be chosen in this case to reduce the effects of δA?

Note that the controller gains have to be chosen as a function of A, and not of $A+\delta A$, as the control engineer does not know that the system gains will change during operation.

(i) Case $H_y = 0$, $H_r = \frac{a}{Ak_p}$.

$$
\begin{aligned}
y_{ss} &= \frac{k_p(A+\delta A)H_r}{a} r_o \\
&= \frac{k_p(A+\delta A)\frac{a}{Ak_p}}{a} r_o \\
&= \frac{A+\delta A}{A} r_o \\
&= \left(1 + \frac{\delta A}{A}\right) r_o
\end{aligned}
$$

So we see that if there is a 5% change in A, we get a 5% change in output speed.

(ii) Case $H_y \neq 0$, $H_r = \frac{a+k_pAH_y}{k_pA}$.

$$
\begin{aligned}
y_{ss} &= \frac{k_p(A+\delta A)H_r}{a+k_p(A+\delta A)H_y} r_o \\
&= \frac{k_p(A+\delta A)\frac{a+k_pAH_y}{k_pA}}{a+k_p(A+\delta A)H_y} r_o \\
&= \frac{(A+\delta A)(a+k_pAH_y)}{A(a+k_p(A+\delta A)H_y)} r_o \\
&= \left(1 + \frac{a\delta A}{A(a+k_pA(A+\delta A)H_y)}\right) r_o
\end{aligned}
$$

So we see that the sensitivity of the steady-state velocity to changes A is

$$\frac{a}{a + k_p A(A + \delta A)H_y} \frac{\delta A}{A}$$

which can be made small by choosing the proportional gain k_p large.

Prob 3 | The open-loop transfer function of a unity feedback system (i.e., a system whose controller is a proportional gain equal to one) is

$$G(s) = \frac{K}{s(s+5)}$$

The desired system response to a step input is specified as having a peak time less than $t_p = 2$sec and an overshoot less than $M_p = 10\%$.

a) Determine whether both specifications can be met simultaneously by selecting the right value of K.

Closed-loop transfer function

$$T(s) = \frac{K}{s^2 + 5s + K} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

From which we see that

$$\omega_n^2 = K \quad \rightarrow \quad \omega_n = \sqrt{K}$$

$$5 = 2\zeta\omega_n \quad \rightarrow \quad \zeta = \frac{5}{2\sqrt{K}}$$

Overshoot of 10% gives:

$$\zeta = \frac{5}{2\sqrt{K}} \geq -\frac{\ln 0.1}{\sqrt{\ln 0.1^2 + \pi^2}} = 0.6 \quad \rightarrow \quad K \leq 17.36$$

Peak-time of 2 seconds gives:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\sqrt{K}\sqrt{1 - \frac{25}{4K}}} = \frac{2\pi}{\sqrt{4K - 25}} \leq 2$$

$$K \geq \frac{\pi^2 + 25}{4} = 8.72$$

Therefore both conditions can be met if $8.72 \leq K \leq 17.36$.

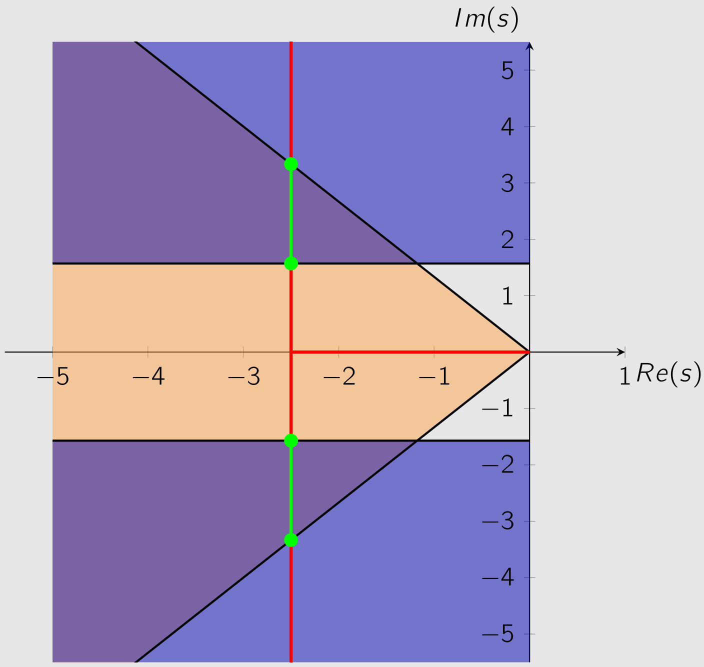
b) Sketch the associated region in the s-plane where both specifications are met, and indicate what root locations are possible for some likely values of K.

We can plot the constraints on the damping ratio and the natural frequency in the shaded region in the figure below.

The closed-loop poles can be calculated as a function of K by solving the characteristic equation

$$s^2 + 5s + K = 0 \qquad \to \qquad s = \frac{-5 \pm \sqrt{25 - 4K}}{2} = -2.5 \pm \sqrt{6.25 - 4K}$$
$$= \begin{cases} -2.5 \pm \sqrt{6.25 - K} & K \leq 6.25 \\ -2.5 \pm j\sqrt{K - 6.25} & K \geq 6.25 \end{cases}$$

The resulting curve is shown in red on the figure below, with the portion satisfying the constraints shown in green ($8.72 \leq K \leq 17.36$).



c) Find the maximum value for K for the system to oscillate.

The system oscillates if the damping ratio is less than one.

$$\omega_n = \sqrt{K}$$
$$5 = 2\zeta\omega_n \qquad \Rightarrow \qquad 1 \geq \zeta = \frac{2.5}{\omega_n} = \frac{2.5}{\sqrt{K}} \qquad \Rightarrow \qquad K \geq 6.25$$